NEWSVENDOR PROBLEM OF A MONOPOLIST WITH CLEARANCE MARKETS

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The classical Newsvendor Problem (NB) considers the procurement decision of a perishable product over a single period model. The per unit revenue, procurement, underage and overage costs are linear. Facing stochastic demand, the objective is to find the optimal procurement quantity so as to maximize the expected profit. It was shown in the literature that if the pricing decision is incorporated into the Newsvendor Problem, the expected profit function loses its concavity property, but can be reduced to a single variable function. Moreover, with mild assumptions on the demand distribution, the objective function has at most two stationary points. In this paper, we extend the NB by assuming that there exists a clearance market at the end of the period. The demand for this market follows any known discrete distribution and it is assumed to be independent of the regular period demand. We prove that the expected profit function of this problem is unimodal. Furthermore, we prove that the expected profit, procurement quantity, and optimal regular price increase in the existence of a clearance market.
Keywords: Newsvendor problem, pricing, clearance market.

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1. Introduction

The classical Newsvendor Problem (NB) considers the single-period procurement decision of a single product that faces a stochastic demand. The procurement, holding, and shortage costs are all linear and the objective is to determine the order quantity that maximizes the single-period expected profit. Arrow, Harris and Marschal (1951) appear to be the first to have studied this problem in some detail. It is well known that the objective function $P^{\mathrm{NB}}(Q)$ is concave with respect to the procurement quantity Q (see Nahmias (1996)). If c_o is the per unit overage cost and c_u is the per unit underage cost, the optimal Q^{NB} is calculated by the well known critical fractile formula $F(Q^{\mathrm{NB}}) = c_u/(c_u + c_o)$.

The NB has been used not only to model the procurement processes in fashion and sporting industries, both at the manufacturing and retail levels (Nahmias (1996)), but also in managing capacity and evaluating advanced booking of orders in service industries such as airlines and hotels, see Netessine et al. (2002), Weatherford and Pfeifer (1994), and Van Mieghem (1998). There have been several extensions of this problem in the literature. Some of these extensions include: different objectives and utility functions, different states of information about demand, multiple products with substitution, random yields, multiple locations, and different pricing strategies. For an extensive review see Khouja (1999).

Although we refer the readers to Khouja (1999) for a thorough review of the extensions of the NB, here we briefly review the papers that incorporate pricing decisions into the NB. As in most inventory models, there is the assumption that the newsvendor can not alter the price, and consequently, the demand pattern. This assumption models the situation in perfect competition markets where price is not decided by any individual vendor. However, under imperfect competition, the vendors can exhibit some monopolistic behavior and influence the demand structure by setting prices. In this situation, we assume that the newsvendor can set the price, and faces a demand level governed by a continuous probability distribution whose expected value is a decreasing function of the price. Under such a setting, the problem is to determine the procurement and pricing strategies that maximize the expected profit over a single period. We call this problem the Newsvendor Problem with Pricing (NBP).

A number of special cases of the NBP have been studied in the literature. The way to capture the effect of the price on the demand structure has been one of the distinctive features of these studies. In the *additive model*, $D(r) = m(r) + \epsilon$ where $D(r)$ is the demand at price r. The term $m(r)$ is a non-increasing non-negative function of price, and ϵ is a random variable with a known distribution. $m(r)$ can be interpreted as either the mean or the minimum demand level at price r, depending on whether $E[\epsilon] = 0$ or μ, respectively. In the *multiplicative model*, $D(r) = m(r)\epsilon$ with $E[\epsilon] = 1$, and in the *riskless model*, $D(r) = m(r)$, that is the demand is a deterministic non-increasing function of price.

Whitin (1955) is the first to introduce pricing decisions into the inventory control theory in a single-period model in mid 1950s. Towards the end of the 50s and early 60s, Mills (1959), (1962), Karlin and Carr (1962) studied the additive and the multiplicative models in some detail. They derived the necessary conditions for optimality, and under reasonable assumptions showed that in the additive model, the optimal price under uncertainty is less than the riskless price, and in the multiplicative model this relationship is reversed. Note that the riskless price is defined as the optimal price when the demand is represented by the riskless model, $D(r) = m(r)$. Zabel worked on the existence and uniqueness of the optimal solutions for the multiplicative (1970) and the additive (1972) models. Young (1962) also studied similar issues for a unified demand model in which $D(r)$ is given by a combination of the additive and the multiplicative forms.

Lau and Lau (1988) analyze two types of demand-price relationships: First, they consider the additive structure with a normally distributed noise term. They show that for a given price the objective function is unimodal over the procurement quantity, and similarly, for given procurement quantity the objective function is unimodal over the price. Based on these results they develop a procedure to find a local maximum, which they refer to as "the optimal solution" without any proof. In section **??**, we actually show that the local maximum they find is the unique local maximum and hence is indeed the global maximum for this type of demand distributions. The second demand structure they consider is generated by a combination of statistical data analysis and experts' subjective opinions. For this case, they can only develop some numerical procedures to find a local maximum. Lau and Lau further consider the maximization of the probability of achieving a certain profit level by determining optimal ordering and pricing levels for the above two demand structures.

In the early 1990s, Polatoğlu (1991) analyzed the NB for both the additive and the multiplicative demand price relationships. The author assumes that there is an initial inventory of i, the average demand $m(r)$ is a decreasing linear function of r over $(0, \infty)$, and that there exists a fixed ordering cost of K on top of the variable procurement cost c. With these assumptions, the expected profit function is shown to be a unimodal function of the procurement quantity and the price for uniformly distributed additive demand and exponentially distributed multiplicative demand.

The above studies indicate that while the existence of an optimal solution can be shown under restrictive assumptions on $m(r)$, the uniqueness of the optimal solution requires further restrictions, especially on the distribution of ϵ.

The most recent and complete study of the NBP appears to be Petruzzi and Dada (1999). In that paper, the authors study the NBP under very general demand distributions including the popular log-concave family. They consider both the additive and the multiplicative demand-price relationships and show that the objective function can have at most two stationary points in the feasible region. They also derive the conditions under which the objective function is unimodal.

Cachon and Kok (2003) consider the NB with a clearance market. The price during the period is given but at the end of the period the excess inventory is sold in a secondary market (clearance market) with a different price. Besides deciding the optimal procurement quantity that is to be sold during the regular period, they also decide the clearance market price which has to be less than the regular period price. This study basically relaxes the assumption that salvage value is a known parameter, and treats it as a decision variable that can be used to balance the demand and supply more efficiently. The regular period demand is stochastic but the second period demand (i.e. demand from the clearance market) is a deterministic function of the excess inventory at the end of the first period, and the clearance market price. For the demand structure they assume, it is shown that the objective function is concave, and the optimal procurement quantity and the salvage value can be determined by solving the first order necessary conditions.

This study considers the additive demand-price relationship, and can be thought of as an extension to both Petruzzi and Dada (1999), and Cachon and Kok (2003). It is an extension to the Petruzzi and Dada (1999), because we assume that there is a clearance market, which has a stochastic demand for the end of period excess inventory that is on sale for a fixed price r_s. This category's demand is assumed to follow a known distribution that is independent of the regular customer demand. This study is an extension to Cachon and Kok (2003), because in this model although we assume that the second category price r_s is fixed, the price charged to the regular customers r is a decision variable. Furthermore, the demand of the clearance market is not deterministic, but rather a general discrete random variable. In the next two sections we first formulate the newsvendor problem of a monopolist with a clearance market, and then solve it. In the Economical Implications of the Clearance Market section, we study the mathematical and economic effects of having a second customer pool. All these results are demonstrated with a numerical study in section 5. We conclude the paper and discuss further research avenues in the last section.

2. Problem Formulation

At the beginning of the period, the newsvendor makes the procurement decision and sets the price for the regular customers. After satisfying the demand of the regular customers, the excess inventory at the end of the period is sold to the second category customers in the clearance market. Regular demand is modeled as $D = a - br + \epsilon$, where r is the regular per unit price and ϵ is the random term which is assumed to follow a known continuous distribution F and density function f over $[0, \Delta]$. The per unit holding and shortage costs associated with regular demand are h and s, respectively. Clearance market demand D_s is stochastic and follows a known general discrete demand distribution $F_s(x)$ over $\{d_1, d_2, \ldots, d_n\}$. For all $i = 1, 2, \ldots, n-1$, the possible demand values $d_i \in \Re^+$, and are ordered as $d_i < d_{i+1}$. The corresponding probability mass function $f_s(x)$ is

$$f_s(\epsilon_s = d_j) = q_j \text{ for all } d_j \in \Re^+, \; j = 1, 2, ..., n, \; 0 \leq q_j \leq 1, \text{ and } \sum_{j=1}^{n} q_j = 1.$$

The above scenario can be the situation for major providers of perishable products and/or services, which have regular demand coming from an infinite pool and at the end of the period, there is a small group of retailers (or any type of customers) that order in bulk and are charged a discounted price. These customers might wait till the end of the period because of the discounted price. In such a problem setting, while regular demand is well estimated by a continuous distribution, the second category demand is better represented by a discrete distribution. Consider the apparel industry. Retailers face a continuous demand for the seasonal fashion goods during the season, but at the end of the season the excess inventory has to be cleared, very likely at a price much lower than the regular season price. This can be done by bulk sales to a number of smaller retailers. In another context, consider the bakeries; they produce and sell baked goods to public during the day facing a constant stream of customers that creates a continuous demand distribution. At the end of the day, the excess stock might be sold to farmers as animal food.

The clearance market price is known to be r_s, and it is assumed to be less than the regular period price. Clearance market sales take place at the end of the period. After satisfying this second category demand, any excess inventory is assumed to have no salvage value and is disposed without paying any extra holding cost. The newsvendor does not have to satisfy the demand of the clearance market customers, and hence it is assumed that there is no shortage cost for the unsatisfied second category demand.

For the above described setting, the newsvendor's objective is to maximize the single period expected profit, $P(u, r)$, by determining the optimal procurement quantity and the regular period price. We call this problem the Newsvendor Problem of a Monopolist with a Clearance Market (NBM), and it is modelled as a nonlinear program

$$\max_{u,r} \{P(u, r) \text{ s.t. } 0 \leq u \leq \Delta + d_n, \text{ and } \underline{r} \leq r \leq \bar{r}\}. \tag{1}$$

The upper bound $u \leq \Delta + d_n$ is included in the formulation, because the optimal solution never procures more than the sum of the maximum demands of the two customer categories.

The objective function $P(u, r)$ assumes two types of functional forms depending on whether any product is specifically procured for the clearance market or not. That is, whether $u > \Delta$ or not. If $u > \Delta$, then the amount $u - \Delta$ is specifically procured and carried in stock to satisfy the demand in the clearance market. This is a case that can take place if the price for clearance market customers is high enough to pay the procurement and the holding cost. So, for $u > \Delta$, $u - \Delta$ units are procured specifically to satisfy the clearance market demand, and the remaining Δ units are procured *mainly* to satisfy the regular customer demand. We said "Δ units are procured *mainly* to satisfy the regular customer demand" instead of "Δ units are *specifically* procured to satisfy the regular customer demand", because in case there is excess inventory from the regular customers and excess demand from the clearance market, this excess stock can be used to satisfy the clearance market demand. We call this phenomena "demand substitution", and denote the expected amount of substitution given u, by $E_s[S|u]$. With this intuition, the objective function $P(u,r)$ for $u > \Delta$ is the sum of three terms: two newsvendor profit functions for the two demand types, and the expected benefit received from substitution. The corresponding newsvendor profit function of each customer category for the given procurement quantities Δ, and $u-\Delta$ are

$$\begin{aligned} P^{\mathrm{NBP}}(\Delta, r) &= (a - br + \mu)(r - c) - \Lambda(\Delta)(c+h) \\ P_s^{\mathrm{NB}}(u-\Delta, r_s) &= \mu_s (r_s - c) - \Lambda_s(u-\Delta)(c+h) - \Theta_s(u-\Delta)(r_s - c). \end{aligned}$$

The subscript s in the second function indicates that the expectation is taken over the clearance market demand distribution. For the regular customers there is no shortage cost because procurement level Δ is the maximum possible demand level. Using these arguments, the objective function $P(u,r)$ for $u > \Delta$ is

$$P(u,r|u>\Delta) = P^{\mathrm{NBP}}(\Delta, r) + P_s^{\mathrm{NB}}(u-\Delta, r_s) + E_s(S|u)(r_s + h).$$

Because of the discrete nature of the clearance market demand distribution, the calculation of $\Lambda_s(u-\Delta)$, $\Theta_s(u-\Delta)$, and $E_s(S|u)$ depend on the relative magnitude of $u-\Delta$ as compared to the possible demand points d_i. To be more specific, if we define d_0 to be 0, and i is such that $d_{i-1} \le u - \Delta \le d_i$, then

$$\begin{aligned} \Lambda_s(u-\Delta) &= \sum_{j=1}^{i-1} q_j (u - \Delta - d_j) \\ \Theta_s(u-\Delta) &= \sum_{j=i}^{n} q_j (\Delta + d_j - u) \\ E_s(S|u) &= \sum_{j=i}^{n} q_j \left\{ \int_{x=0}^{[u-d_j]^+} (d_j + \Delta - u)\, dF(x) + \int_{x=[u-d_j]^+}^{\Delta} (\Delta - x)\, dF(x) \right\}. \end{aligned}$$

So,

$$P(u,r|u>\Delta) = P_i(u,r) \text{ if } \Delta + d_{i-1} \le u \le \Delta + d_i, \text{ and } \underline{r} \le r \le \bar{r}, \text{ for any } i = 1, 2, \ldots, n,$$

where

$$\begin{aligned} P_i(u,r) = {} & \mu_s (r_s - c) - (c+h) \sum_{j=1}^{i-1} q_j (u - \Delta - d_j) - (r_s - c) \sum_{j=i}^{n} q_j (\Delta + d_j - u) \\ & + (a - br + \mu)(r - c) - (c+h)\Lambda(\Delta) + \sum_{j=i}^{n} q_j \left(\int_{x=0}^{[u-d_j]^+} (d_j + \Delta - u)\, dF(x) \right. \\ & \left. + \int_{x=[u-d_j]^+}^{\Delta} (\Delta - x)\, dF(x) \right) (r_s + h), \text{ for } i = 1, 2, \ldots, n \end{aligned} \tag{2}$$

If $u \le \Delta$, clearance market demand can be satisfied only if there is excess inventory after satisfying the regular customer demand in full. As in the previous case, the objective function is the sum of the total expected profit generated through satisfying the demand of each customer categories. Since there is no product procured specifically for the clearance market, the expected amount of the clearance market demand satisfied is determined by the expected substitution term defined above. So, besides the expected profit from the regular customers, which is described by the objective function of the NBP, $P^{\mathrm{NBP}}(u,r)$, we also have the expected profit from the clearance market, which is calculated as $E_s(S|u)(r_s + h)$. Hence the objective function $P(u,r)$ for $u \le \Delta$, defined as $P_0(u,r)$, is

$$P_0(u,r) = P^{\mathrm{NB}}(u,r) + E_s(S|u)(r_s + h).$$

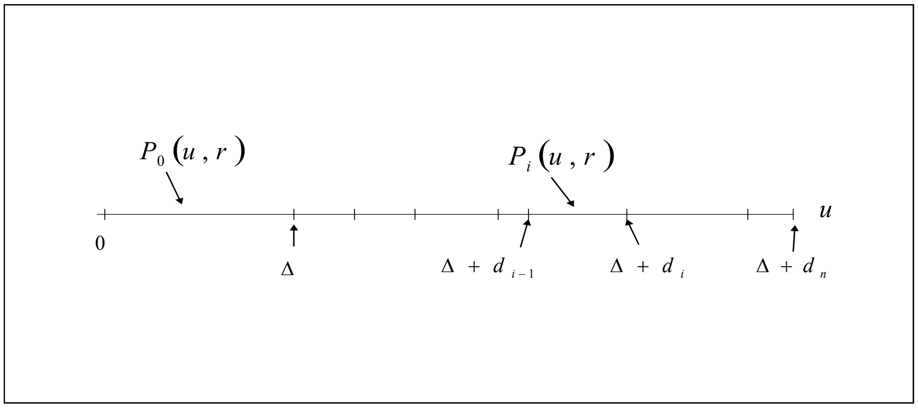

Figure 1: Different functional forms of $P(u,r)$ over $0 \le u \le \Delta + d_n$

More explicitly,

$$\begin{aligned} P_0(u,r) &= (a - br + \mu)(r - c) - (c+h)\Lambda(u) - (r+s-c)\Theta(u) \\ &+ \sum_{j=1}^{n} q_j \left(\int_{x=0}^{[u-d_j]^+} d_j dF(x) + \int_{x=[u-d_j]^+}^{u} (u-x)\, dF(x) \right)(r_s + h). \end{aligned} \tag{3}$$

The expected total single period profit from the two customer categories, is

$$P(u,r) = \begin{cases} P_0(u,r), & \text{if } u \le \Delta \\ P_i(u,r), & \text{if } \Delta + d_{i-1} \le u \le \Delta + d_i, \ i = 1, 2, \ldots, n. \end{cases}$$

Figure 1 illustrates the range $P_i(u,r)$ $i = 0, 1, \ldots, n$ is defined over.

In the next section, we analyze the objective function $P(u,r)$ in more detail, reveal its concavity properties, and determine the optimal solution under a fairly weak assumption on the regular customer demand distribution $F(\cdot)$.

3. Analysis and the Solution

The objective function of the NBM is at least as complicated as the objective function of the NBP. As expected, it is neither convex nor concave over the entire feasible region (see Karakul (2004) for a proof.) Hence, we can not necessarily determine the optimal solution by solving the first order necessary conditions. It is not easy to characterize how many local maxima such an analysis might generate, and hence we need to resort to other techniques. The next lemma reveals an important property of the objective function, which helps us transform the objective function to a function of a single variable.

Lemma 1 *The objective function $P(u,r)$ is concave in any of its arguments if the other is fixed.*

Proof. Proof is mechanical and can be found in Karakul (2004). ■

Since $P^{\text{NBM}}(u,r)$ is concave with respect to r given u, the optimal unconstrained price as a function of u has to satisfy the first order necessary condition $\partial P(u,r)/\partial r = 0$. Using this property, the following lemma determines the optimal bounded price as a function of u.

Lemma 2 *For any given $u \in [0, \Delta + d_n]$, the optimal bounded price for the regular customers is a non-decreasing function of u, and is determined as:*

$$r^{NBM}(u) = \begin{cases} \underline{r} & \text{for } r(u) \le \underline{r} \\ r(u) & \text{for } \underline{r} \le r(u) \le \bar{r} \\ \bar{r} & \text{for } \bar{r} \le r(u) \end{cases}$$

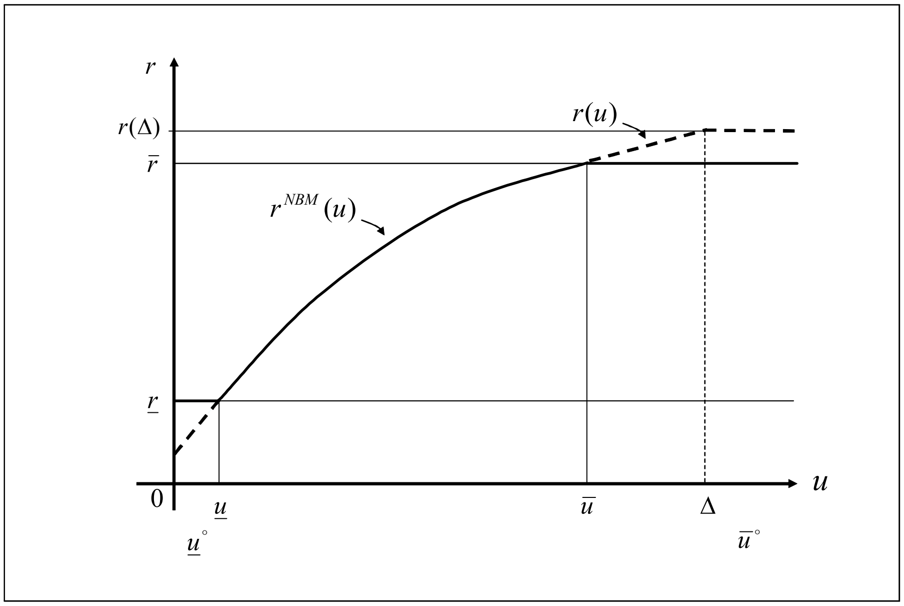

Figure 2: The optimal regular category price as a function of u

where

$$r(u) = \begin{cases} \frac{a+bc+\mu-\Theta(u)}{2b}, & \textit{for } 0 \le u \le \Delta \\ \frac{a+bc+\mu}{2b}, & \textit{for } \Delta \le u \le \Delta + d_n \end{cases}$$

Proof. Proof is in Karakul (2004). ■

The objective function is concave in r for any given u, and $r(u)$ is a monotonic non-decreasing function of u, and hence there exists unique values $\underline{u} \le \bar{u}$ such that $r(u)$ equals $\underline{r}$ and $\bar{r}$, respectively. Hence

$$r^{\mathrm{NBM}}(u) = \begin{cases} \underline{r} & \text{for } u \le \underline{u} \\ r(u) & \text{for } \underline{u} \le u \le \bar{u} \\ \bar{r} & \text{for } \bar{u} \le u. \end{cases}$$

If we assume that there exists $\underline{u} \le \bar{u}$ such that $0 \le \underline{u} \le \bar{u} \le \Delta$, then the optimal price for the regular customers can be depicted as in Figure 2

An important observation from the above lemma is that the optimal regular customer price, $r^{\mathrm{NBM}}(u)$, is not affected from the quantity procured specifically for the clearance market. For $u \ge \Delta$ it is equal to $\max((a + bc + \mu)/2b, \bar{r})$, a constant. This is an intuitive result because the price of the regular customers is assumed to have no effect on the demand of the clearance market. With this intuition the concavity of $P(u, r)$ over $u \ge \Delta$ becomes clearer. If the optimal procurement quantity is more than the maximum possible demand of the regular customers, the problem becomes very similar to the newsvendor problem that considers only the clearance market customers. The only difference is that some of the excess demand for the clearance market can be satisfied from the first period excess inventory. That is, up to "$\Delta -$ actual demand of the regular customers" items procured for the regular customers can be used to satisfy the clearance market demand in case of a shortage.

Substituting $r^{\mathrm{NBM}}(u)$ into the objective function $P(u, r)$, the nonlinear program in 1 is transformed to a one-dimensional optimization over u :

$$\max P(u)$$
$$\text{s.t. } 0 \le u \le \Delta + d_n$$

where

$$P(u) = \begin{cases} \underline{P}(u) = P_0(u, \underline{r}), \text{for } 0 \le u \le \underline{u} \\ P_0(u) = P_0(u, r(u)), \text{ for } \underline{u} \le u \le \bar{u} \\ \bar{P}_0(u) = P_0(u, \bar{r}), \text{ for } \bar{u} \le u \le \Delta \\ \bar{P}_i(u) = P_i(u, \bar{r}), \text{ for } \Delta + d_{i-1} \le u \le \Delta + d_i, \ i = 1, \ldots, n \end{cases}$$

In the objective function $P(u)$, the optimal price is set to $\underline{r}$ for $u \le \underline{u}$, $r(u)$ for $\underline{u} \le u \le \bar{u}$, and $\bar{r}$ for $\bar{u} \le u \le \Delta + d_n$. By Lemma 1, the objective function $P(u, r)$ is concave in u for given r. So, if we define $\underline{u}^*$ and $\bar{u}^*$ as the maximizers of $P(u)$ over $0 \le u \le \underline{u}$ and $\bar{u} \le u \le \Delta + d_n$, respectively, then $\underline{u}^*$ and $\bar{u}^*$ are determined through marginal analysis. In theory, the way to obtain these values are the same, but because of the $n+1$ functions defined over the range $\bar{u} \le u \le \Delta + d_n$, calculation of $\bar{u}^*$ is more involved than that of $\underline{u}^*$, To determine $\underline{u}^*$ we first calculate the unconstrained local maximum $\underline{u}^\circ$ of $\underline{P}(u)$ from the first order necessary condition

$$\frac{d\underline{P}(u)}{du} = (1 - F(u))(\underline{r} + s + h) - (c + h) + \sum_{j=1}^{n} q_j \left(F(u) - F(u - d_j)\right)(r_s + h) = 0.$$

Depending on the magnitude of $\underline{u}^\circ$, the optimal $\underline{u}^*$ is determined as

$$\underline{u}^* = \begin{cases} 0, \text{ if } \underline{u}^\circ \le 0 \\ \min(\underline{u}, \underline{u}^\circ), \text{ if } \underline{u}^\circ > 0 \end{cases}$$

Over the range $\bar{u} \le u \le \Delta + d_n$ the optimal price is constant and the objective function is concave, hence $\bar{u}^*$ is efficiently obtained by determining the smallest $i = 0, 1, \ldots, n$ such that

$$\frac{d\bar{P}_i(u)}{du}\Big|_{u=\Delta+d_i} < 0, \text{ where } d_0 = 0,$$

and solving $d\bar{P}_i(u)/du = 0$ for $\bar{u}^\circ \in [\Delta + d_{i-1}, \Delta + d_i]$. Then, $\bar{u}^* = \bar{u}^\circ$.

For $\underline{u} \le u \le \bar{u}$, it is not clear how many local maxima might exist (see Figure 3, next page). Although line search methods are easy to implement and generally very quick to find the local maxima of single variable nonlinear functions, there are functions which are so ill-behaved that even this task might be very time consuming. To make sure that this is not the case for $P(u)$ over $\underline{u} \le u \le \bar{u}$, the next section introduces a condition such that $P_0(u)$ is either unimodal or monotonic over $\underline{u} \le u \le \bar{u}$ if $F(\cdot)$ satisfies this condition. This result, together with some further properties of the objective function $P(u)$ at the boundaries, help prove that the objective function of the NBM is a unimodal function.

3.1 A Unimodality Condition for the Objective Function

In this section, it is assumed that the clearance market demand follows any general discrete distribution, and the demand distribution of the regular customers satisfies the following condition:

$$2z(u)^2 + dz(u)/du \ge G(u) \text{ for all } 0 \le u \le \Delta \tag{4}$$

where

$$z(u) = f(u)/\left(1 - F(u)\right),$$

is the hazard rate of the regular customer demand distribution $F(\cdot)$, and

$$G(u) = \frac{2b\,(r_s + h)\sum_{j=0}^{n} q_j\,[f'(u)f(u - d_j) - f(u)f'(u - d_j)]}{\left(1 - F(u)\right)^3}$$

All log-concave distributions satisfy this condition, because they have increasing hazard rates, i.e. $z'(u) \ge 0$, and for such distributions $f'(u)f(u - d^j) \le f(u)f'(u - d^j)$ making $G(u) \le 0$, see M.Y. An (1995). Log-concave distributions include the well-known Normal, Log-Normal, Uniform, Exponential, Gamma, etc..

For this class of distributions, it can be shown that $P(u)$ is a unimodal function over $[0, \Delta + d_n]$. But, first we need the following result regarding $P_0(u)$.

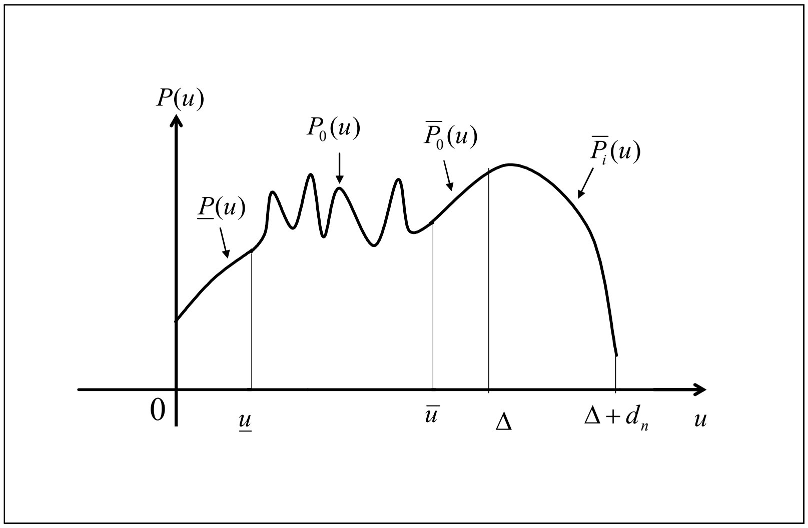

Figure 3: Possible shape of $P(u)$

Theorem 3 $P_0(u)$ *is either unimodal or monotonic non-decreasing over* $0 \leq u \leq \Delta$.

Proof. First show that there exists at most one stationary point of $P_0(u)$ over $[0, \Delta]$. From the chain rule:

$$\frac{dP_0(u)}{du} = (1 - F(u))\,(r(u) + s + h) - (c + h) + \sum_{j=1}^{n} q_j\,(F(u) - F(u - d_j))\,(r_s + h) \tag{5}$$

Let $P_{0u}(u)$ and $f'(u)$ respectively represent $dP_0(u)/du$ and $df(u)/du$. To find the number and characteristics of the zeros of $P_{0u}(u)$, consider its first and second derivatives

$$\begin{aligned} \frac{dP_{0u}(u)}{du} &= -\frac{f(u)}{2b}\left\{2b\,[r(u) + s + h] - \frac{1 - F(u)}{z(u)}\right\} + (r_s + h)\sum_{j=1}^{n} q_j\,(f(u) - f(u - d_j)) \\ \frac{d^2 P_{0u}(u)}{du^2} &= -f'(u)\frac{1}{2b}\left\{2b\,[r(u) + s + h] - \frac{1 - F(u)}{z(u)}\right\} - \frac{f(u)\,(1 - F(u))}{2bz(u)^2}\left\{2z(u)^2 + z'(u)\right\} \\ &\quad + (r_s + h)\sum_{j=1}^{n} q_j\,(f'(u) - f'(u - d_j)) \end{aligned}$$

The second derivative of $P_{0u}(u)$ at any of its stationary point is

$$\begin{aligned} \frac{d^2 P_{0u}(u)}{du_2^2} \quad | \quad {}_{dP_{0u}/du=0} &= (r_s + h)\sum_{j=1}^{n} \frac{q_j}{f(u)}\,[f'(u)f(u - d_j) - f(u)f'(u - d_j)] \\ &\quad - \frac{f(u)\,(1 - F(u))}{2bz(u)^2}\left\{2z(u)^2 + z'(u)\right\} \end{aligned}$$

Hence, if condition 4 holds, that is,

$$2z\,(u)^2 + z'\,(u) \geq \frac{2b\,(r_s + h)\sum_{j=1}^{n} q_j\,[f'\,(u)\,f\,(u - d_j) - f\,(u)\,f'\,(u - d_j)]}{(1 - F(u))^3}$$

for all $0 \leq u \leq \Delta$, then at any stationary point of $P_{0u}(u)$ we have $d^2 P_{0u}(u)/du^2 \leq 0$ proving that there can be at most one stationary point of $P_{0u}(u)$, and if it exists it is a local maximum. This implies that

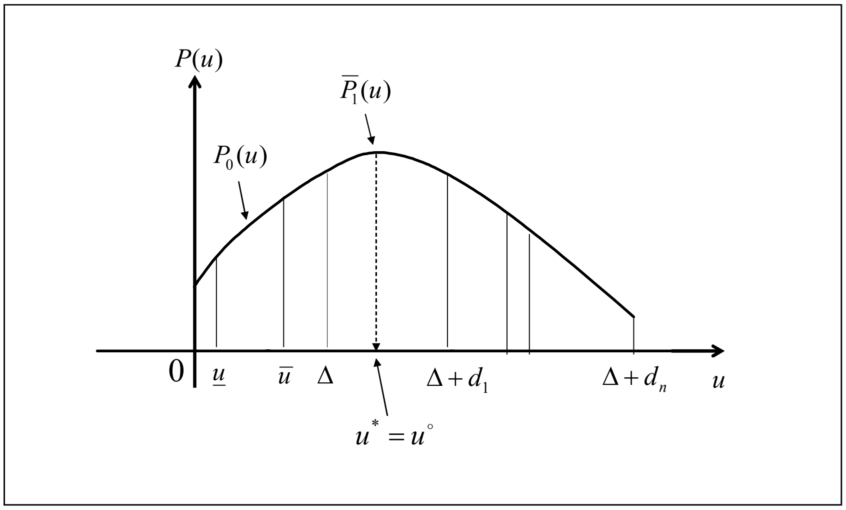

Figure 4: An illustration of the objective function $P(u)$ when $F(\cdot)$ belongs to the family of distributions discussed in this section.

$P_{0u}(u) = 0$ can have at most two roots over $[0, \Delta]$ and consequently $P_0(u)$ might possibly have two stationary points. But, note that

$$P_{0u}(0) = r(0) + s - c > 0.$$

This holds because by Lemma 2,

$$r(0) = \frac{a + bc}{2b} = \frac{c}{2} + \frac{a}{2b}.$$

In order for the problem to have a feasible price range, i.e.

$$\{r : \underline{r} = \max(c, r_s) \leq r \leq \bar{r} = \min(a/b, r_u)\}$$

is not an empty set, it is necessary to have $a/b > c$. Hence, $r(0) > c$, and since $s \geq 0$ we have $r(0) + s - c > 0$. So, $P_{0u}(u)$ has exactly one root if $P_{0u}(\Delta) \leq 0$ and it is a local maximum of $P_0(u)$ because the sign of $P_{0u}(u)$ changes from positive to negative. If $P_{0u}(\Delta) \geq 0$, then $P_{0u}(u)$ has no root in which case $P_0(u)$ is a non-decreasing function over $0 \leq u \leq \Delta$. This completes the proof. ■

Using the above results:

Theorem 4 *$P(u)$ is a unimodal function over $0 \leq u \leq \Delta + d_n$.*

Proof. From Lemma 1, $P(u)$ is concave over $[0,\underline{u}]$ and $[\bar{u}, \Delta + d_n]$, and from Theorem 3, $P_0(u)$ is unimodal over $[\underline{u},\bar{u}]$. Since $P(u)$ is continuously differentiable, it is unimodal. This completes the proof. ■

An immediate corollary to this theorem is that only one of the functions $\underline{P}(u)$, $P_0(u)$, and $\bar{P}_i(u)$ $i = 0, 1, \ldots, n$ has a feasible stationary point over the range it is defined. This function and consequently the unique feasible stationary point, which is a local maximum, can be easily determined by checking the sign of the slopes of these functions at the boundaries of the regions they are defined over. There exists only one function which is increasing at the lower boundary and decreasing at the upper boundary, and the local maximum of this function, call it u°, is feasible and it is the optimum u^*. See Figure 4 for an illustration.

The optimal procurement quantity and the regular customer price is given as follows:

Corollary 5 *Let u^* be the unique feasible local maximum of $P(u)$ over $0 \leq u \leq \Delta + d_n$. The optimal procurement quantity Q^{NBM} and the regular period price r^{NBM} for the NBM are, respectively, $m(r^{NBM}) + u^*$ and $r^{NBM}(u^*)$.*

Proof. Proof immediately follows from the definitions of Q, r^{NBM}, and u^*. ■

For the demand distributions considered in this section, the above discussion proves that solving the newsvendor problem of a monopolist with a clearance market is no more difficult than solving the newsvendor problem with pricing.

The next section focuses on the managerial insights, and answers: What kind of economical implications does having a clearance market have?

4. Economic Implications of Having The Second Customer Category

The unconstrained optimal price for the regular customers is $r(u) = (a + bc + \mu - \Theta(u))/2b$ regardless of the existence of the clearance market. This is exactly the unconstrained optimal price of the NBP, where there exists no clearance market, see Karakul (2004). For both problems NBP and NBM, $Q = a - br(u) + u$ is a strictly increasing function of u because

$$\frac{dQ(u)}{du} = -b\frac{dr(u)}{du} + 1 = -\frac{(1 - F(u))}{2} + 1 > 0,$$

By Lemma 6, $Q^{\text{NBM}} \geq Q^{\text{NBP}}$ and hence $u^{\text{NBM}} \geq u^{\text{NBP}}$. Since $r(u)$ is a non-decreasing function of u, we have $r(u^{\text{NBM}}) \geq r(u^{\text{NBP}})$ for $u^{\text{NBM}} \geq u^{\text{NBP}}$. So, not only the optimal procurement quantity but also the optimal price of the regular customers is increased with the introduction of the clearance market. Having a clearance market, we procuree more, and increase the service level of the regular customers. As a result of this increased service level, a premium of $r^{\text{NBM}} - r^{\text{NBP}}$ is charged to the regular customers.

Q, which is defined as $Q = a - br(u) + u$, and r^{NBM} are non-decreasing functions of u. At first glance this result seems to be contradictory. It sounds like while the price is increased, the procurement quantity also increases, which normally should decrease because increased price decreases the demand. This argument is true, but in our case both Q and r increase as a result of an increase in a third variable u. Increased u has the effect of increasing the price $r(u)$ and this has an indirect effect of decreasing Q by decreasing the minimum demand level $m(r)$. But, this effect is not large enough to make Q a decreasing function of u. Under a different demand price relationship, for example the multiplicative demand case, this might not be true and Q might be decreasing in u. This has to be studied further. If u is kept constant and price is increased, the optimal procurement quantity decreases as expected.

We present the next lemma to demonstrate the relationship between the optimal procurement quantities of the three models:1) NB, the classical newsvendor problem where price is exogeneous; 2) NBP, the newsvendor problem with pricing; and 3) NBM, the newsvendor problem with pricing and a clearance market.

Lemma 6 *Define* $\left(Q^{\text{NBP}}, r^{\text{NBP}}\right)$ *as the newsvendor solution when there is no clearance market, and* Q_s^{NB} *as the classical newsvendor solution when the demand distribution is* F_s *and the parameters are* r_s, c, *and* h. *At optimality we have* $Q^{\text{NBP}} \leq Q^{\text{NBM}} \leq m(r^{\text{NBM}}) + \Delta + Q_s^{\text{NB}}$.

Proof. For $Q^{\text{NBM}} \leq \Delta$, it is only necessary to show that $Q^{\text{NBP}} \leq Q^{\text{NBM}}$, which is equivalent to showing $u^{\text{NBP}} \leq u^{\text{NBM}}$. Since $u^{\text{NBP}} \leq \Delta$, it is necessary to show that $P_0(u)$ is non-decreasing for $u \leq u^{\text{NBP}}$. From equation 5

$$\frac{dP_0(u)}{du} = (1 - F(u))\,(r(u) + s + h) - (c + h) + \sum_{j=1}^{n} q_j\,(F(u) - F(u - d_j))\,(r_s + h)\,.$$

We have

$$(1 - F(u))\,(r(u) + s + h) - (c + h) \geq 0,$$

because

$$\frac{dP^{\text{NBP}}(u)}{du} = (1 - F(u))\,(r(u) + s + h) - (c + h)$$

and $P^{\text{NBP}}(u)$ is unimodal over $[0, \Delta]$ with mode u^{NBP}. Furthermore, $F(u) \geq F(u - d_j)$ for every u and $j = 1, 2, ..., n$. Hence, $dP_0(u)/du \geq 0$ for all $0 \leq u \leq u^{\text{NBP}}$.

Now, let $Q^{\mathrm{NBM}} \geq \Delta$. For this case, it is necessary to show that for all $u \geq \Delta + Q_s^{\mathrm{NB}}$, $P(u)$ is non-increasing. Let $Q_s^{\mathrm{NB}} = d_{k-1}$ be the classical newsvendor solution for the second customer category. Then, k is calculated as the smallest i for which

$$\sum_{j=i}^{n} q_j\,(r_s + h) - (c+h) \leq 0$$

For $u \geq \Delta + Q_s^{\mathrm{NB}}$, $P(u)$ is equal to $\bar{P}_i(u)$ if $u \in [\Delta + d_{i-1}, \Delta + d_i]$ $i = k, k+1, \ldots, n$. Without loss of generality let $\Delta + d_{i-1} \leq u \leq \Delta + d_i$. The first derivative of $\bar{P}_i(u)$ is

$$\frac{d\bar{P}_i(u)}{du} = \sum_{j=i}^{n} q_j\,(r_s+h) - (c+h) - \sum_{j=i}^{n} q_j F(u-d_j)\,(r_s+h)\,.$$

For all $i \geq k$, since

$$\sum_{j=i}^{n} q_j\,(r_s+h) - (c+h) \leq 0, \text{ and } F(u-d_j) \geq 0,$$

$d\bar{P}_i(u)/du \leq 0$, proving that $P(u)$ is non-increasing over $[\Delta + d_{i-1}, \Delta + d_i]$. Since i is arbitrary, $P(u)$ is non-increasing over $[\Delta + Q_s^{\mathrm{NB}}, \Delta + d_n]$. This completes the proof. ■

Intuitively, an expected result of risk pooling is that the optimal procurement quantity to be less than or equal to the sum of the two independent newsvendor procurement quantities, Q^{NBP} and Q_s^{NB}. This is not necessarily true. Sometimes the price for the clearance market might be so low that even though we have sure demand from this category, it is not profitable to procure any products specifically for them. In this case, the independent newsvendor procurement quantity for this category is zero. When there are two categories of customers, the procurement quantity is greater than or equal to the independent newsvendor quantity of the regular customers. This happens because the excess inventory at the end of the period after fully satisfying the demand of the regular customers can be used to satisfy the demand of the clearance market. Hence, the combined expected marginal benefit from the first and the second category customers might be larger than the expected marginal loss incurred due to the higher procurement quantity Q^{NBM} (which is greater than or equal to Q^{NBP}).

In summary, a newsvendor that has two customer categories procures more and sets a higher price as compared to the one who has only the regular customers.

5. Numerical Analysis

In this section we conduct a numerical study to demonstrate the possible benefits that can be realized through incorporating pricing decisions and clearance markets to models that arise in a newsvendor setting.

Let the regular period demand be $D = a - br + \epsilon$, and ϵ follows a truncated gamma distribution F over $[0, 250]$ with scale parameter $\gamma = 2$ and shape parameter $\beta = 30$. This distribution belongs to the log-concave family, and it has a positively skewed unimodal shape, see Figure 5.

Although there is no restriction on the distribution of the clearance market, for the purposes of this numerical study, it is assumed to be a triangular discrete distribution. The demand values and the respective probabilities are as follows:

$$\begin{aligned} D_s &= d_i \text{ with probability } p_i \;\; i = 1, 2, \ldots, 5 \\ d_1 &= 50,\; d_2 = 100,\; d_3 = 150,\; d_4 = 200,\; d_5 = 250 \\ p_1 &= \frac{1}{9},\; p_2 = \frac{2}{9},\; p_3 = \frac{3}{9},\; p_4 = \frac{2}{9},\; p_5 = \frac{1}{9} \end{aligned}$$

Furthermore, let the demand and the cost parameters assume the following values for the base case:

$$\begin{array}{lllll} a = 1000 & b = 30 & \Delta = 250 & r_u = 30 & \bar{r} = \min(\frac{a}{b}, r_u) = 30 \\ c = 10 & s = 15 & h = 4 & r_s = 13 & \underline{r} = \max(c, r_s) = 13. \end{array}$$

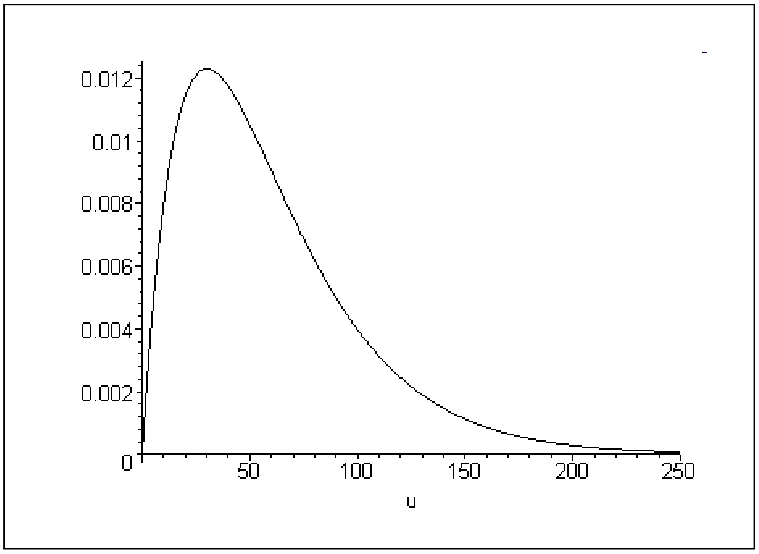

Figure 5: The shape of $F(x) \sim Gamma(2,30)$ truncated at $x = 250$.

End of period excess inventory is assumed to be sold at the clearance market for r_s dollars per unit. For any product sold in the clearance market not only a revenue of r_s dollars is received but also the holding cost h is saved. Hence, a retailer that has a clearance market for her excess inventory procures more than one who does not have a clearance market. See the $\Delta Q\%$ columns of Tables 1 to 4. Depending on the parameter values, the procurement quantity in the presence of a clearance market, Q^{NBM}, is approximately 6% to 30% more than Q^{NBP}, the procurement quantity when there exists no clearance market.

Table 1 demonstrates the sensitivity of the profit and the procurement quantity to the salvage value r_s. We consider $r_s = \alpha c$, for $\alpha = 0.5, 0.6, \ldots, 1.5$. The clearance market price has no effect on the NBP model, which does not consider a clearance market. Hence, $(Q^{\text{NBP}}, P^{\text{NBP}})$ does not change with different r_s values. As α, consequently r_s, increases the clearance market becomes more and more profitable, which increases the optimal procurement quantity and the profit generated by using the NBM model. See columns headed Q^{NBM}, and P^{NBM}. If r_s is at least as high as the procurement cost c, that is $\alpha = 1$, the percentage increase in the profit is 12.9% and the percentage increase in the procurement quantity is 14.11%. The final observation from this table is that, having the clearance market increases the optimal price ($r^{\text{NBM}} \geq r^{\text{NBP}}$), but the difference is quite low, less than 1%, see column $\Delta r\%$, where

$$\Delta r\% = 100\frac{r^{\text{NBM}} - r^{\text{NBP}}}{r^{\text{NBP}}}.$$

Table 2 shows how the solutions of each model, NBP, and NBM are effected from the price elasticity b. We solve each model for $b = 5, 10, ..., 50$. As the elasticity increases, the price, procurement quantity and profit of each model decrease. For each value of $b = 5, 10, ..., 50$, NBM generates more profit, and procures more than NBP, and the percentage differences $\Delta P\%$ and $\Delta Q\%$ are increasing in b. For $b = 5, 10, 15, 20$ the optimal price suggested by each model is the same and equal to the upper bound r_u. This happens because the demand is not very sensitive to price and both pricing models suggest a higher price than r_u.

Next two tables, Tables 3 and 4 analyze the sensitivity of the NBP and NBM models to the holding cost h, and the shortage cost s. From Table 3 we observe that the optimal price, procurement quantity and profit of both models decrease as h increases from 4 to 13. This agrees with the intuition that if holding cost is high, we tend to keep less stock on hand.

In Table 4 we vary the shortage cost s from 50% to 150% of the procurement cost c. As s increases having shortage becomes very expensive and hence it is necessary to procure more. Both NBP and NBM agree on this observation. As u increases, both the optimal price and the optimal procurement quantity increase, demonstrating the discussion in the economic implications section. The benefit of clearance market

α	u^{NBM}	r^{NBM}	Q^{NBM}	P^{NBM}	u^{NBP}	r^{NBP}	Q^{NBP}	P^{NBP}	$\Delta P\%$	$\Delta Q\%$	$\Delta r\%$
0.5	96	22.56	420	4421.49	68	22.44	395	4155.51	6.40	6.34	0.52
0.6	102	22.57	425	4465.40	68	22.44	395	4155.51	7.46	7.64	0.58
0.7	108	22.59	430	4513.98	68	22.44	395	4155.51	8.63	9.08	0.64
0.8	115	22.60	437	4567.62	68	22.44	395	4155.51	9.92	10.63	0.70
0.9	121	22.61	443	4626.66	68	22.44	395	4155.51	11.34	12.30	0.75
1.0	129	22.62	450	4691.44	68	22.44	395	4155.51	12.90	14.11	0.79
1.1	137	22.63	458	4762.39	68	22.44	395	4155.51	14.60	16.11	0.83
1.2	146	22.64	467	4840.02	68	22.44	395	4155.51	16.47	18.36	0.86
1.3	155	22.64	476	4924.87	68	22.44	395	4155.51	18.51	20.66	0.88
1.4	162	22.65	483	5015.67	68	22.44	395	4155.51	20.70	22.38	0.90
1.5	168	22.65	489	5111.00	68	22.44	395	4155.51	22.99	23.85	0.91

Table 1: Sensitivity of the NBM model to the salvage value

b	u^{NBM}	r^{NBM}	Q^{NBM}	P^{NBM}	u^{NBP}	r^{NBP}	Q^{NBP}	P^{NBP}	$\Delta P\%$	$\Delta Q\%$	$\Delta r\%$
5	158	30.00	1008	18301.31	75	30.00	925	17451.66	4.87	8.93	0.00
10	158	30.00	858	15301.31	75	30.00	775	14451.66	5.88	10.66	0.00
15	158	30.00	708	12301.31	75	30.00	625	11451.66	7.42	13.22	0.00
20	158	30.00	558	9301.31	75	30.00	475	8451.66	10.05	17.40	0.00
25	157	26.17	502	6667.55	71	25.96	422	5858.34	13.81	18.87	0.83
30	155	22.64	476	4924.87	68	22.44	395	4155.51	18.51	20.66	0.88
35	154	20.12	450	3715.90	65	19.94	367	2977.50	24.80	22.55	0.92
40	154	18.23	424	2840.47	63	18.06	341	2126.88	33.55	24.63	0.94
45	153	16.76	399	2187.39	61	16.60	314	1494.13	46.40	26.97	0.95
50	153	15.58	373	1689.95	60	15.44	288	1013.68	66.71	29.66	0.96

Table 2: The effect of price elasticity on the NBM and NBP models

h	u^{NBM}	r^{NBM}	Q^{NBM}	P^{NBM}	u^{NBP}	r^{NBP}	Q^{NBP}	P^{NBP}	$\Delta P\%$	$\Delta Q\%$	$\Delta r\%$
4	155	22.64	476	4924.87	68	22.44	395	4155.51	18.51	20.66	0.88
5	153	22.64	474	4916.49	66	22.43	393	4134.90	18.90	20.57	0.93
6	151	22.64	471	4908.62	64	22.42	391	4115.52	19.27	20.43	0.97
7	148	22.64	469	4901.24	62	22.41	390	4097.23	19.62	20.23	1.01
8	146	22.64	467	4894.33	61	22.40	389	4079.93	19.96	20.07	1.05
9	144	22.64	465	4887.83	59	22.39	388	4063.55	20.28	19.93	1.09
10	142	22.63	463	4881.68	58	22.38	387	4047.99	20.60	19.80	1.12
11	141	22.63	462	4875.85	57	22.37	386	4033.20	20.89	19.69	1.16
12	139	22.63	460	4870.30	56	22.36	385	4019.11	21.18	19.60	1.19
13	138	22.63	459	4865.00	54	22.36	384	4005.66	21.45	19.51	1.23

Table 3: Sensitivity of the NBM and NBP models to the holding cost

s	u^{NBM}	r^{NBM}	Q^{NBM}	P^{NBM}	u^{NBP}	r^{NBP}	Q^{NBP}	P^{NBP}	$\Delta P\%$	$\Delta Q\%$	$\Delta r\%$
15	155	22.64	476	4924.87	68	22.44	395	4155.51	18.51	20.66	0.88
14	155	22.64	476	4925.81	67	22.44	394	4168.53	18.17	20.84	0.91
13	155	22.64	475	4926.75	66	22.43	393	4181.93	17.81	21.02	0.94
12	154	22.64	475	4927.71	65	22.43	392	4195.72	17.45	21.22	0.97
11	154	22.64	475	4928.68	63	22.42	391	4209.92	17.07	21.41	1.00
10	154	22.64	474	4929.66	62	22.41	390	4224.57	16.69	21.62	1.03
9	153	22.64	474	4930.66	61	22.40	389	4239.69	16.30	21.83	1.07
8	153	22.64	473	4931.68	60	22.39	388	4255.31	15.89	22.04	1.11
7	152	22.64	473	4932.70	58	22.38	387	4271.46	15.48	22.26	1.15
6	152	22.64	472	4933.75	57	22.37	386	4288.19	15.05	22.48	1.19
5	151	22.64	472	4934.81	56	22.36	385	4305.53	14.62	22.71	1.24

Table 4: Sensitivity of the NBM and NBP models to the shortage cost

increases as the shortage cost increases, see the column $\Delta P\%$ for an increasing percentage profit increase in the existence of the clearance market. As in the above instances, considering a clearance market increases the price as well, but the magnitude of the change is still marginal, approximately 1%.

6. Conclusion and Future Research Avenues

This paper focused on the newsvendor problem with pricing when the demand price relationship is well represented by the additive model, and the excess inventory can be salvaged at a clearance market at the end of the period. Existence of a clearance market makes the objective function assume two types of functional forms, depending on whether we procure any products specifically for the clearance market or not. We show that although the objective function over the entire feasible region is neither convex nor concave with respect to both variables, it is concave with respect to either of the variables if the other one is given. Using this property it is shown that the optimal price can be determined as a non-decreasing function of the procurement quantity on top of the minimum demand level, and hence the two-variable optimization problem can be transformed to a single-variable optimization problem. Under very general assumptions on the demand distribution of the regular customers, it is shown that the objective function is unimodal. As a result of our analysis, we prove that having a clearance market increases not only the procurement quantity, but also the price for the regular customers. This is quite intuitive, because now any excess inventory after fully satisfying the demand of the regular customers can be used to satisfy the clearance market demand at the end of the period. Clearance market brings us more flexibility, and hence we can procure more and charge more to the regular customers. Through an experimental study, it is demonstrated that the procurement quantity is increased about 10% for a wide range of parameter values, but the increase in price seems to be quite low, around 1% level.

There are two important conclusions one can come to from this paper: The first one is regarding the level of theoretical difficulty attached to integrating pricing decisions and clearance markets to the NB, and how this difficulty can be efficiently dealt with the approach developed and applied in this paper. We have demonstrated that even for the simplest stochastic inventory control problem, integrating pricing decisions make the objective function lose its concavity property. What is encouraging is that the objective function is concave with respect to any of the variables if the other one is fixed, and the optimal unconstrained price can be explicitly expressed as a function of the procurement quantity. Using these properties the two-variable nonlinear optimization problem can be transformed into a single-variable optimization. For a large family of demand distributions, this single-variable objective function can be shown to be unimodal, with a unique mode that can be determined through marginal analysis. This approach and the successful results make us believe that similar analysis can be carried out for multiple products.

The second important conclusion is regarding the practical benefits that can be achieved through these models. Integrating pricing decision and considering a secondary market, the company not only increases its profit but also increases the procurement level and the price for the regular customers, best of all worlds.

There are several extensions to the above models that can be considered. In this study, the price for the clearance market is assumed to be given; making this price a decision variable might increase the profitability of the firm even more. As a second extension, the multiplicative demand-price relationship can be considered. This is a very important extension because for some real life scenarios the additive model might not give satisfactory results. Extending the model to multiple products and multiple periods would be very desirable considering that there are a great number of real life scenarios that involve multiple products over multiple periods. Finally, the continuous approach has to be converted to discrete analysis when we are dealing with discrete unit of products, and the demand patterns of the customers follow discrete distributions. Our initial analysis regarding the discrete analysis have shown that if the demand distributions satisfy some more restrictive conditions, results similar to the ones in this paper can be achieved.

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