EFFECT OF SAMPLE SIZE AND DISTRIBUTION PARAMETERS ON CONFIDENCE LOWER BOUNDS FOR WEIBULL PERCENTILES

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Abstract: The Weibull distribution is one of the most widely used lifetime distribution in reliability analysis. Estimating lower percentiles of the Weibull distribution has been a major concern, because the estimates are unreliable when the sample size is small. Small sample sizes are common for long and expensive experiments such as the ones for modeling fracture strength of composite materials. Therefore, several recent studies focused on constructing confidence lower bounds on the lower percentiles; a comparison of methods for estimating these bounds showed that the maximum likelihood method has the best performance. In this work, this method is used for assessing the effect of sample size and distribution parameters on the bounds. A new pivotal statistic is defined and the bounds are shown to be a function of this statistic and the sample size. After a Monte-Carlo simulation, its behaviour as a function of the sample size is modeled by non-linear regression analysis. Approximate functional expressions are proposed for the distance between the bounds and the true percentiles; the first and tenth percentiles are chosen for this purpose. These functions can be easily used and interpreted by an experimenter.

Keywords: *Reliability, Weibull Distribution, Monte-Carlo Simulation*

WEIBULL YÜZDELİKLERİNİN GÜVEN ALT SINIRLARINA ÖRNEK HACMİ VE DAĞILIM PARAMETRELERİNİN ETKİSİ

Özet: Weibull dağılımı, güvenilirlik analizinde en yaygın kullanılan hayat süresi dağılımlarından biridir. Weibull dağılımının alt yüzdeliklerinin tahmini önemli bir problemdir, çünkü örnek hacmi küçükse tahminler güvenilmezdir. Kompozit malzemelerin kopma mukavemetinin ölçümü gibi uzun ve pahalı deneylerde küçük örnek hacimlerine sıkça rastlanır. Bu yüzden, son zamanlarda pek çok çalışma alt yüzdelikler için alt güven sınırları geliştirilmesine odaklanmıştır; bu sınırları tahmin eden yöntemler kıyaslanmış ve maksimum benzerlik yönteminin en iyi performansı verdiği görülmüştür. Bu çalışmada, bu yöntem kullanılarak örnek hacmi ve dağılım parametrelerinin sınırlara etkileri incelenmektedir. Yeni bir pivotal istatistik tanımlanarak sınırların bu istatistiğin ve örnek hacminin bir fonksiyonu olduğu gösterilmiştir. İstatistiğin örnek hacmine bağlı davranışı doğrusal olmayan regresyon analiziyle modellenmiştir. Gerçek yüzdeliklerle sınırlar arasındaki uzaklık için yaklaşık fonksiyonlar önerilmiş, bu amaçla birinci ve onuncu yüzdelikler seçilmiştir. Bu fonksiyonlar deney yapanlar tarafından kolayca kullanılabilir ve yorumlanabilir.

Anahtar Kelimeler: *Güvenilirlik, Weibull Dağılımı, Monte-Carlo Benzetimi*

1. Introduction

The Weibull distribution is a very popular lifetime distribution in reliability analysis. Also it has been widely used to model fracture strength of composite materials. Because high experimental costs limits the size of experiments, estimating confidence lower bounds for Weibull percentiles has drawn a great deal of attention in materials science literature (Barbero et al., 2000; Birgoren, 2003).

Following the notation of materials science, the two-parameter Weibull distribution function is given by $F(\sigma)=1\text{-}\exp(\text{-}(\sigma/\sigma_0)^m)$, where σ denotes the variable tensile strength, m is the shape parameter, and σ_0 is the scale parameter. The parameters m and σ_0 are estimated from a sample of strength measurements $\sigma_1, \sigma_2, \ldots, \sigma_n$; let's denote the estimates by $\hat{m}$ and $\hat{\sigma}_0$. By setting this function to a probability *p*, the (100*p*)th percentile σ_p is obtained as $\sigma_p=\sigma_0(\text{-}\ln(1\text{-}p))^{1/m}$. By placing $\hat{m}$ and $\hat{\sigma}_0$ in place of *m* and σ_0, the estimate of σ_p, $\hat{\sigma}_p$ is obtained. However, these values are unreliable for small samples. Therefore, confidence lower bounds for σ_p have been used to characterise a material. A comparison of methods for estimating confidence lower bounds showed that the maximum likelihood method is better than different variations of the weighted linear regression methods (Birgoren, 2003). Therefore, this method is chosen as the method of parameter estimation in this study. The procedure for estimating $\hat{m}$ and $\hat{\sigma}_0$ by this method is given by Law and Kelton (2001).

When maximum likelihood estimates are used, the statistic $\hat{m}\ln\left(\hat{\sigma}_p/\sigma_p\right)$ is distributed independently of m and σ_0, hence a pivotal statistic (Fernandez-Saez et al., 1993). By specifying a c_p value such that $Pr(\hat{m}\ln\left(\hat{\sigma}_p/\sigma_p\right) \le c_p) = 1-\alpha$, the lower bound for $(1-\alpha)$ level one-sided confidence intervals for σ_p is derived as $l_p = \hat{\sigma}_p \exp\left(-c_p/\hat{m}\right)$, ie $Pr(l_p \le \sigma_p) = 1-\alpha$. The l_p values for $\alpha = 0.05$ and $p = 0.01$ and $p = 0.10$ are called the A-basis and B-basis material properties, respectively.

This study aims to model the closeness between σ_p and l_p: the ratio $\left(\sigma_p - l_p\right)/\sigma_p$ can be used as a measure of precision. It measures the percent departure of l_p from the true percentile σ_p.

2. Simulation and Parameter Fitting

The bound l_p can be reformulated as $l_p = \sigma_p \exp(S_p/m)$, where $S_p = (\hat{m}\ln(\hat{\sigma}_p/\sigma_p) - c_p)/(\hat{m}/m)$ is a pivotal statistic. Let's denote the $5th$ percentile of l_p by $l_{p(0.05)}$, which satisfies $Pr(l_p \le l_{p(0.05)}) = 0.05$. l_p is an increasing function of S_p. Then, $l_{p(0.05)}$ can be computed from the $5th$ percentile of S_p, $S_{p(0.05)}$: $l_{p(0.05)} = \sigma_p \exp\left(S_{p(0.05)}/m\right)$. Therefore $Pr(l_{p(0.05)} \le l_p \le \sigma_p) = 0.90$.

In order to model the behaviour of $l_{p(0.05)}$, a multi-stage Monte-Carlo simulation study was performed. In the first part, the c_p values were estimated following the simulation procedure of Birgoren (2003) by setting $\alpha = 0.05$ and $N = 1$ .000.000. This simulation was performed for all combinations of $p = 0.01$ and 0.10, and $n = 6, 8, 10, 15, 20, ..., 50, 60, 70, ..., 100$.

The second part estimates $S_{p(0.05)}$ using the simulated c_p values: N Weibull samples of size n were generated with $m = 1$ and $\sigma_0 = 1$; for each sample, the S_p value was estimated. Using N simulated values of S_p, provides an estimate of $S_{p(0.05)}$. This procedure was repeated for all combinations of p and n. The estimated $S_{p(0.05)}$ values are not presented here for space reasons.

Using non-linear regression analysis, the $S_{p(0.05)}$ values were fitted as a function of n by

$$S_{p(0.05)} = b_1 - b_2 * \exp\left(\left(n - b_3\right)^{-b_4}\right) \tag{1}$$

The empirical parameters $b_1, ..., b_4$ for $p = 0.01$ and 0.10 are presented in Table 1. When compared to the estimated $S_{p(0.05)}$ values, the maximum error of the empirical function in Equation 1 is 0.4% for both $p = 0.01$ and $p = 0.10$. It seems to be a very good fit.

Using $l_{p(0.05)} = \sigma_p \exp\left(S_{p(0.05)}/m\right)$, the ratio $\left(\sigma_p - l_{p(0.05)}\right)/\sigma_p$ can be modeled as

$$\left(\sigma_p - l_{p(0.05)}\right)/\sigma_p = \left[1 - \exp\left(\left\{b_1 - b_2 * \exp\left(\left(n - b_3\right)^{-b_4}\right)\right\}/m\right)\right] \tag{2}$$

It can be shown that

$$Pr\left[0 \le \left(\sigma_p - l_p\right)/\sigma_p \le \left(\sigma_p - l_{p(0.05)}\right)/\sigma_p\right] = 0.90 \tag{3}$$

Table 1. Parameters for the empirical function for $S_{p(0.05)}$

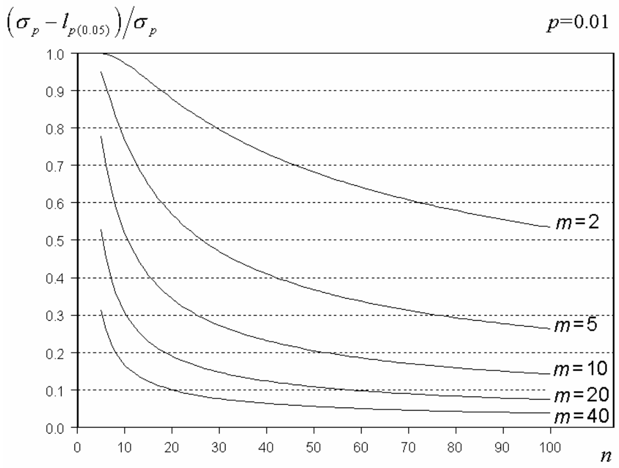
	b_1	b_2	b_3	b_4
$p = 0.01$	23.57	23.8582	1.9259	0.6502
$p = 0.10$	12.0175	12.1435	1.8909	0.6171

Therefore, $\left(\sigma_p - l_{p(0.05)}\right)/\sigma_p$ is a probabilistic upper bound for $\left(\sigma_p - l_p\right)/\sigma_p$, which is the precision measure for l_p in this study.

3. Effect of Parameters and Sample Size

A detailed examination of Equations 2 and 3 helps to give a comprehensive perspective about the effect of parameters and sample size on the precision of the confidence lower bounds. The percent departure of the lower bounds from the true percentile is variable from sample to sample. Equation 2 forms an upper bound for this random quantity with a high probability. Figure 1 plots the upper bound with respect to n for $p = 0.01$; m is varied in a wide range from 2 to 40. It is clear from the plot that the function is decreasing in n and m.

The plot shows that marginal change in the function values are much more significant when $n \leq 30$. It also points out achievable levels of precision: for a wide range of m values, 100 measurements will suffice to reduce the percent error of A-basis material properties below 60%. Finally, it should be noted that Figure 3 can be easily reproduced for $p = 0.10$ and also for any m and n via a spreadsheet program.



4. Conclusion

This work is focused on the maximum likelihood estimation of the confidence lower bounds for the two-parameter Weibull distribution. It proposes probabilistic upper bounds on the deviations of the lower bounds from the true percentiles. They are expressed in the form of two formulae, which are functions of the sample size and the Weibull modulus. The upper bound levels are plotted for various sample sizes and m values. For samples sizes less then 30, the plot exhibit much higher downward slope, which indicates much higher gains in precision with the same amount of additional experimental costs.

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